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EVOLUTION OF EHRESMANN'S JET THEORY

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1. Introduction.

Jets of mappings introduced by Ehresmann [1] are still the most useful objects for formulating geometric frameworks of physical theories. We are proposing modifications designed to make jet theory less dependent on local coordinates. Extensions of the theory with applications to the calculus of variations and mechanics are also proposed.

2. The p^k vitesses and points proches.

The p^k *vitesses* in a manifold N were originally defined as equivalence classes of differentiable mappings from $\mathbb{R}^p$ to N. Mappings

$$\gamma : \mathbb{R}^p \to N$$

and

$$\gamma' : \mathbb{R}^p \to N$$

are equivalent if

$$\partial_{\boldsymbol{i}}(f \circ \gamma')(0) = \partial_{\boldsymbol{i}}(f \circ \gamma)(0)$$

for each function $f : N \to \mathbb{R}$ and each multi-index $\boldsymbol{i} \in \mathbb{N}^p$ with length $|\boldsymbol{i}| = \boldsymbol{i}_1 + \cdots + \boldsymbol{i}_p \leqslant k$. Equivalence classes are the p^k vitesses. Ehresmann's construction extended the definition of the tangent vector of a curve to higher dimensions an higher differential orders.

Defining a vector as a derivation was the approach preferred by some mathematicians. André Weil's response [11] to Ehresmann's construction was a generalization of the concept of a derivation to higher differential orders based on local algebras. Let A be a local

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algebra. A *point proche* in a manifold N is a unit preserving morphism

$$u : C^\infty(N) \to A,$$

where $C^\infty(N)$ is the algebra of differentiable functions on N.

Weil claimed that his construction was more general. My agreement with this statement is not unqualified. Here are some comments.

(1) It is true that vitesses can be obtained as points proches. Let $\mathsf{I}_0(\mathbb{R}^p, 0)$ be the set of differentiable functions on $\mathbb{R}^p$ vanishing at $0 \in \mathbb{R}^p$. This set is a maximal ideal in the algebra $C^\infty(\mathbb{R}^p)$ of differentiable functions on $\mathbb{R}^p$. Let $\mathsf{I}_k(\mathbb{R}^p, 0)$ be the power $(\mathsf{I}_0(\mathbb{R}^p, 0))^{k+1}$. The quotient

$$\mathsf{A}^k(\mathbb{R}^p, 0) = C^\infty(\mathbb{R}^p) \big/ \mathsf{I}_k(\mathbb{R}^p, 0)$$

is a local algebra. The construction of points proches in N associated with this algebra reproduces the construction of the p^k vitesses in N.

(2) The construction of p^k vitesses extends to the definition of k-jets of mappings

$$\gamma : M \to N.$$

Pairs (γ, x) of a mapping γ and a point $x \in M$ are classified. Pairs (γ, x) and (γ', x') are equivalent if $x' = x$ and

$$\partial_{\boldsymbol{i}}(f \circ \gamma' \circ \xi^{-1})(0) = \partial_{\boldsymbol{i}}(f \circ \gamma \circ \xi^{-1})(0)$$

for each function $f : N \to \mathbb{R}$, each chart

$$\xi : M \to \mathbb{R}^p$$

defined in a neighbourhood of x, and each multi-index $\boldsymbol{i} \in \mathbb{N}^p$ with length $|\boldsymbol{i}| = \boldsymbol{i}_1 + \cdots \boldsymbol{i}_p \leqslant k$. The equivalence class is the k-jet of γ at x. Only jets of mappings with a distinguished source point can be produced as points proches. Weil's approach would have to be generalized to make it applicable to more general jets.

(3) In applications to differential geometry jets are equivalence classes of mappings. Simple generalizations of Ehresman's construction seem to provide all applicable types of jets. The generality of Weil's abstract approach is in my opinion excessive.

3. Borrowing from Weil the algebraic definition of derivatives.

3.1. Ideals in the algebras of differentiable functions.

With a differential manifold M we associate the algebra $C^\infty(M)$ of differentiable functions on M.

We will denote by $\mathbb{K}$ the set $\mathbb{N} \cup \{\infty, \mathfrak{c}\}$, where $\mathfrak{c}$ stands for the cardinality of $\mathbb{R}$. The ordering relations $\leqslant$, $<$, $\geqslant$, and $>$ have in $\mathbb{K}$ the usual meaning of inequalities of cardinal numbers. Let x be a point in a differential manifold M. In the algebra $C^\infty(M)$ of differentiable functions on M we introduce a sequence of ideals

$$\mathsf{I}_0(M, x),\ \mathsf{I}_1(M, x),\ \ldots,\ \mathsf{I}_\infty(M, x),\ \mathsf{I}_\mathfrak{c}(M, x).$$

The ideal $\mathsf{I}_0(M, x)$ associated with x is maximal in the sense that it is not a proper subset of any ideal except the trivial ideal $C^\infty(M)$. It is known that all maximal ideals in $C^\infty(M)$ are associated with points.

For $k \in \mathbb{N}$, the ideal $\mathsf{I}_k(M,x)$ is the power $(\mathsf{I}_0(M,x))^{k+1}$ of the ideal $\mathsf{I}_0(M,x)$. The ideal $\mathsf{I}_\infty(M,x)$ is the intersection $\bigcap_{k\in\mathbb{N}} \mathsf{I}_k(M,x)$. The ideal $\mathsf{I}_\mathfrak{c}(M,x)$ is the set of functions each vanishing in a closed neighbourhood of x. Inclusion relations

$$\mathsf{I}_k(M,x) \subset \mathsf{I}_{k'}(M,x)$$

hold for all k' and k in $\mathbb{K}$ such that $k' \leqslant k$.

3.2. Jets and germs of mappings.

Let $C^\infty(N|M)$ denote the space of differentiable mappings from a differential manifold M to a differential manifold N. In the set $C^\infty(N|M) \times M$ we introduce an equivalence relation for each $k \in \mathbb{K}$. Two pairs (φ, x) and (φ', x') are equivalent if $x' = x$ and

$$g \circ \varphi' - g \circ \varphi \in \mathsf{I}_k(M,x)$$

for each $g \in C^\infty(N)$. The equivalence class of (φ, x) is denoted by $\mathsf{j}^k\varphi(x)$ and is called the *k-jet* of φ at x. A $\mathfrak{c}$-jet is also called a *germ*. The set of k-jets is denoted by $\mathsf{J}^k(N|M)$. The *source* and *target projections* are the mappings

$$\sigma_{k(N|M)} : \mathsf{J}^k(N|M) \to M : \mathsf{j}^k\varphi(x) \mapsto x$$

and

$$\tau_{k(N|M)} : \mathsf{J}^k(N|M) \to N : \mathsf{j}^k\varphi(x) \mapsto \varphi(x).$$

The symbol $\mathsf{J}^k(N|M,x)$ will denote the fibre

$${\sigma_{k(N|M)}}^{-1}(x) = \left\{ a \in \mathsf{J}^k(N|M);\ \sigma_{k(N|M)}(a) = x \right\}.$$

Inclusion relations

$$\mathsf{I}_k(M,x) \subset \mathsf{I}_{k'}(M,x)$$

for k' and k in $\mathbb{K}$ such that $k' \leqslant k$ imply the existence of canonical epimorphisms

$${\tau^{k'}}_{k(N|M,x)} : \mathsf{J}^k(N|M,x) \to \mathsf{J}^{k'}(N|M,x)$$

for each $x \in M$. Hence, we have epimorphisms

$${\tau^{k'}}_{k(N|M)} : \mathsf{J}^k(N|M) \to \mathsf{J}^{k'}(N|M).$$

Relations

$${\tau^{k''}}_{k(N|M)} = {\tau^{k''}}_{k'(N|M)} \circ {\tau^{k'}}_{k(N|M)}$$

hold for $k'' \leqslant k' \leqslant k$.

Jets of local mappings can be composed. If $\mathsf{j}^k\varphi(x) \in \mathsf{J}^k(N,y|M,x)$ and $\mathsf{j}^k\psi(y) \in \mathsf{J}^k(O,z|N,y)$, then $\mathsf{j}^k\psi(y) \circ \mathsf{j}^k\varphi(x)$ is an element of $\mathsf{J}^k(O,z|M,x)$ defined by

$$\mathsf{j}^k\psi(y) \circ \mathsf{j}^k\varphi(x) = \mathsf{j}^k(\psi \circ \varphi)(x).$$

Sets

$$\mathsf{J}^k(N,y|M,x) = \left\{ a \in \mathsf{J}^k(N|M);\ \sigma_{k(N|M)}(a) = x,\ \tau_{k(N|M)}(a) = y \right\}$$

are used.

The *k-jet prolongation* of a mapping $\varphi : M \to N$ is the mapping

$$\mathsf{j}^k\varphi : M \to \mathsf{J}^k(N|M) : x \mapsto \mathsf{j}^k\varphi(x).$$

Jets of functions can be multiplied. Sets $\mathsf{J}^k(\mathbb{R}|M,x)$ are local algebras isomorphic to the quotient algebras $\mathsf{A}^k(M,x) = C^\infty(M)/\mathsf{I}_k(M,x)$. These algebras could be used to reproduce the p^k vitesses as points proches at each fixed point x.

3.3. Jets and germs of submanifolds and jets with volume.

Jets of submanifolds and subsets in general can be defined. Let (S,x) and (S',x') be pairs composed each of a subset of M and one of its points. The pairs are equivalent if $x' = x$ and

$$\mathsf{I}_k(M,x) + \mathsf{I}_0(M,S') = \mathsf{I}_k(M,x) + \mathsf{I}_0(M,S).$$

This establishes an equivalence relation. The equivalence class of (S,x) is denoted by $\mathsf{j}^k S(x)$ and called the *k-jet* of S at x.

A vector $v \in \mathsf{T}_x M$ is said to be *tangent* to a jet $\mathsf{j}^k S(x) = \mathsf{I}_k(M,x) + \mathsf{I}_0(M,S)$ if $\langle \mathrm{d}f, v\rangle = 0$ for each function $f \in \mathsf{I}_k(M,x) + \mathsf{I}_0(M,S)$. Vectors tangent to a jet form the *tangent set* of the jet. A *k-jet of a set with a q-volume element* is a pair $(\mathsf{j}^k S(x), w)$ composed of a jet $\mathsf{j}^k S(x)$ and a q-vector w in the tangent space of $\mathsf{j}^k S(x)$. We will denote by $\mathsf{V}^{k,q}M$ the space of k-jets of subsets of M with q-volume elements.

3.4. Other interpretations of power ideals.

Consider the lattice of closed subsets of M. Unions and intersections of closed sets are closed. Infinite intersections are still closed. Infinite unions are not necessarily closed. With a closed subset $S \subset M$ we associate the ideal

$$\mathsf{I}_0(M,S) = \left\{ f \in C^\infty(M);\ \forall_{x\in S}\ f(x) = 0 \right\}.$$

The ideal $\mathsf{I}_0(M,M)$ associated with the whole manifold M contains only the zero function. The ideal $\mathsf{I}_0(M,\emptyset)$ is the whole algebra $C^\infty(M)$. These two ideals are considered *trivial*. The following relations hold.

(1) $\mathsf{I}_0(M,S) \subset \mathsf{I}_0(M,S')$ if $S' \subset S$.
(2) $\mathsf{I}_0(M,S\cup S') = \mathsf{I}_0(M,S) \cap \mathsf{I}_0(M,S')$.
(3) $\mathsf{I}_0(M,S\cap S') = \mathsf{I}_0(M,S) + \mathsf{I}_0(M,S')$.

These relations do not extend to infinite unions and intersections. An infinite intersection of ideals associated with closed sets corresponds to the closure of the union of the sets. An infinite union of ideals associated with closed sets does not necessarily correspond to a closed set. Examples will be given below.

Products of ideals associated with sets are not usually associated with sets. Maximal ideals

$$\mathsf{I}_0(M,x) = \mathsf{I}_0(M,\{x\})$$

corresponding to single point sets were introduced earlier. The powers $\mathsf{I}_k(M,x)$ of these ideals are not associated with sets. These power ideals could be considered *enlarged points*.

The ideal $\mathsf{I}_k(M,x) + \mathsf{I}_0(M,S)$ used in the definition of the jet $\mathsf{j}^k S(x)$ of a submanifold can be used to represent the jet. This ideal is not associated with a set.

3.5. The support of a current.

Let $\boldsymbol{c}$ be a current of dimension 0 in a differential manifold M and let

$$Z(\boldsymbol{c}) = \Big\{x \in M;\ \exists_{U \subset M}\ x \in U\ \forall_{f \in C^\infty(M)}\ \operatorname{supp}(f) \subset U \Rightarrow \langle f, \boldsymbol{c}\rangle = 0\Big\}.$$

The *support* of $\boldsymbol{c}$ is the set $\operatorname{supp}(\boldsymbol{c}) = M \backslash Z(\boldsymbol{c})$. This definition is based on Theorem 7 p. 40 of de Rham's book.

A new definition is proposed. The *support* of a current $\boldsymbol{c}$ is the ideal

$$\mathsf{S}(\boldsymbol{c}) = \Big\{f \in C^\infty(M);\ \forall_{g \in C^\infty(M)}\ \langle gf, \boldsymbol{c}\rangle = 0\Big\}.$$

PROPOSITION 1.

$$\operatorname{supp}(\boldsymbol{c}) = \Big\{x \in M;\ \forall_{f \in \mathsf{S}(\boldsymbol{c})}\ f(x) = 0\Big\}.$$

PROOF:

a) Let $x \in Z(\boldsymbol{c})$ and let U be an open neighbourhood of x such that $\langle h, \boldsymbol{c}\rangle = 0$ for each function h on M with $\operatorname{supp}(h) \subset U$. Let f be a function on M such that $f(x) \neq 0$ and $\operatorname{supp}(f) \subset U$. For each function g we have $\operatorname{supp}(gf) \subset U$. Hence, $\langle gf, \boldsymbol{c}\rangle = 0$. It follows that $f \in \mathsf{S}(\boldsymbol{c})$. We conclude that x is not in the set

$$\Big\{x \in M;\ \forall_{f \in \mathsf{S}(\boldsymbol{c})}\ f(x) = 0\Big\} \tag{1}$$

since $f(x) \neq 0$. We have proved the inclusion

$$\operatorname{supp}(\boldsymbol{c}) \supset \Big\{x \in M;\ \forall_{f \in \mathsf{S}(\boldsymbol{c})}\ f(x) = 0\Big\}.$$

b) If x is not in the set (1), then there is a function f such that $f(x) \neq 0$ and $\langle gf, \boldsymbol{c}\rangle = 0$ for each function g. Let U be a neighbourhood of x such that $f(x') \neq 0$ for each $x' \in U$. If h is a function with $\operatorname{supp}(h) \subset U$, then $h = gf$, where g is the function

$$\begin{aligned} g &: M \to \mathbb{R} \\ &: x' \mapsto \begin{cases} h(x')/f(x'), & \text{for } x' \in U \\ 0, & \text{for } x' \notin U \end{cases} \end{aligned}$$

Hence, $\langle h, \boldsymbol{c}\rangle = 0$. It follows that $x \in Z(\boldsymbol{c})$. This proves the inclusion

$$\operatorname{supp}(\boldsymbol{c}) \subset \Big\{x \in M;\ \forall_{f \in \mathsf{S}(\boldsymbol{c})}\ f(x) = 0\Big\}.$$

■

4. Ideals of functions on a product manifold.

Given a system of ideals

$$\mathsf{I}(M, x) \subset \mathsf{I}_0(M, x) \subset C^\infty(M)$$

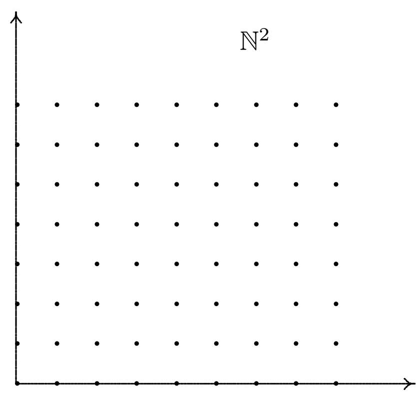
we can introduce generalized jets as equivalence classes of pairs $(\varphi, x) \in C^\infty(N|M) \times M$. Two pairs (φ, x) and (φ', x') are equivalent if $x' = x$ and

$$g \circ \varphi' - g \circ \varphi \in \mathsf{I}(M, x)$$

for each $g \in C^\infty(N)$. If M is a manifold with no additional structure, then the power ideals $\mathsf{I}_k(M, x)$ are the natural choice of ideals of differentiable functions. Other ideals

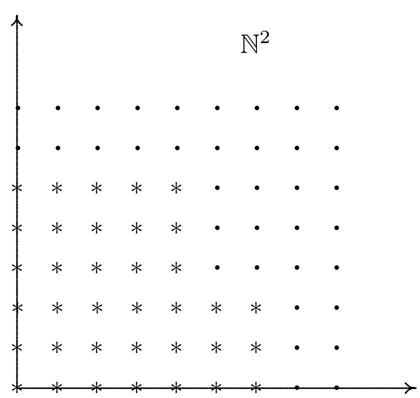
can be constructed in terms of additional structures in M. We will construct ideals of functions on a product manifold.

The set $\mathbb{N}^2$ displayed below



is a lattice. The partial order relation $(k', l') \leqslant (k, l)$ holds if $k' \leqslant k$ and $l' \leqslant l$. A subset $K \subset \mathbb{N}^2$ is an *ideal* if $(k, l) \in K$ and $(k', l') \leqslant (k, l)$ imply $(k', l') \in K$. Here is an example of an ideal:

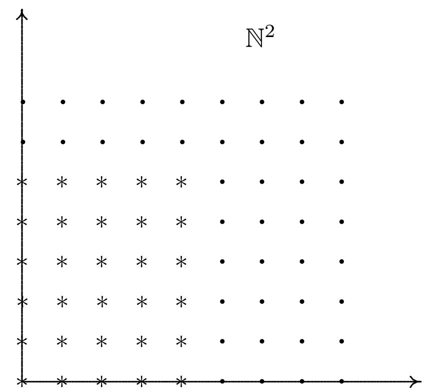
(2)



An element of $\mathbb{N}^2$ generates a *principal ideal*. The ideal generated by (k, l) is the set

$$\left\{(k', l') \in \mathbb{N}^2;\ (k', l') \leqslant (k, l)\right\}$$

The set



is the principal ideal generated by $(4, 5)$. Each ideal is the union of principal ideals. The ideal (2) is the union of principal ideals generated by $(4, 5)$ and $(6, 3)$.

We establish a correspondence between ideals in the algebra $C^\infty(M \times N)$ and the ideals in the lattice $\mathbb{N}^2$. The correspondence associates intersections of ideals in $C^\infty(M \times N)$ with unions of ideals in $\mathbb{N}^2$ and sums of ideals in $C^\infty(M \times N)$ with intersections of ideals in $\mathbb{N}^2$. Inclusions of ideals are reversed by the correspondence. To the principal ideal generated by $(k, l) \in \mathbb{N}^2$ we assign the ideal

$$\mathsf{I}_{(k,l)}(M \times N, (x, y)) = \mathsf{I}_k(M, x) \circ \pi + \mathsf{I}_l(N, y) \circ \rho \subset C^\infty(M \times N)$$

constructed with the canonical projections π and ρ of $M \times N$ onto M and N respectively. This assignment extends to all ideals in $\mathbb{N}^2$ since each ideal is the union of principal ideals.

The construction of ideals of functions on product manifolds produces useful results in the case of the product $\mathbb{R} \times \mathbb{R}$. The 1^l vitesses in a manifold M are the *l-tangent vectors*. These objects are equivalence classes of curves. In the space $\mathsf{T}^l M$ of l-vectors we can construct k-vectors. These objects are equivalence classes of curves in the space of equivalence classes of curves in M. It has been shown [5] that the space $\mathsf{T}^k\mathsf{T}^l M$ can be identified with the space $\mathsf{T}^{(k,l)}M$ of equivalence classes of mappings

$$\chi : \mathbb{R} \times \mathbb{R} \to M.$$

Mappings χ and χ' are equivalent if

$$g \circ \chi' - g \circ \chi \in \mathsf{I}_{(k,l)}(\mathbb{R} \times \mathbb{R}, (0, 0))$$

for each $g \in C^\infty(M)$. This identification makes studying such objects easier.

The set of vitesses in M constructed with an ideal in $C^\infty(\mathbb{R}^2, 0)$ corresponding to an ideal $K \subset \mathbb{N}^2$ will be denoted by $\mathsf{T}^K M$. The equivalence class of a mapping $\chi : \mathbb{R}^2 \to M$ will be denoted by $\mathsf{t}^K\chi(0, 0)$. This set is a differential manifold. Given two ideals K and K' such that $K' \subset K$ we introduce the projection

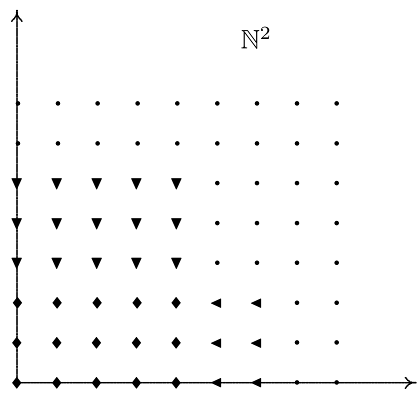
$$\tau^{K'}{}_{K\,M} : \mathsf{T}^K M \to \mathsf{T}^{K'} M : \mathsf{t}^K\chi(0, 0) \mapsto \mathsf{t}^{K'}\chi(0, 0).$$

This projection is a differential fibration.

It was shown in [5] that the manifold $\mathsf{T}^{K\cup K'}M$ is diffeomorphic to the fibre product

$$\mathsf{T}^{K}M \underset{(\tau^{K\cap K'}{}_{K\,M},\,\tau^{K\cap K'}{}_{K'\,M})}{\times} \mathsf{T}^{K'}M.$$

The following diagram displays the union and the intersection of principal ideals generated by $(4,5)$ and $(6,3)$.



5. Iterated tangent functors [4] [5].

It is convenient to associate a covariant functor

$$\begin{array}{c} \mathsf{T}^{k} \\ \Big\downarrow{\scriptstyle \tau_k} \\ \mathsf{I} \end{array}$$

with an index $k \subset \mathbb{N}$. To a manifold M this functor assigns the differential fibration

$$\begin{array}{c} \mathsf{T}^{k}M \\ \Big\downarrow{\scriptstyle \tau_{k\,M}} \\ M \end{array}$$

The differential fibration morphism

$$\begin{array}{ccc} \mathsf{T}^{k}M & \xrightarrow{\mathsf{T}^{k}\varphi} & \mathsf{T}^{k}N \\ \Big\downarrow{\scriptstyle \tau_{k\,M}} & & \Big\downarrow{\scriptstyle \tau_{k\,N}} \\ M & \xrightarrow{\varphi} & N \end{array}$$

is assigned to a morphism $\varphi\colon M \to N$.

The differential fibration morphism

(3)
$$\begin{tikzcd} \mathsf{T}^k\mathsf{T}^l M \arrow[r, "\mathsf{T}^k\tau_{lM}"] \arrow[d, "\tau_{k\mathsf{T}^l M}"'] & \mathsf{T}^k M \arrow[d, "\tau_{kM}"] \\ \mathsf{T}^l M \arrow[r, "\tau_{lM}"] & M \end{tikzcd}$$

is the result of the functor

$$\begin{tikzcd} \mathsf{T}^k \arrow[d, "\tau_k"'] \\ \mathsf{I} \end{tikzcd}$$

applied to the fibration

$$\begin{tikzcd} \mathsf{T}^l M \arrow[d, "\tau_{lM}"'] \\ M \end{tikzcd} .$$

It is convenient to think about the fibration (3) as the double fibration obtained by applying to M the functor

$$\begin{tikzcd} & \mathsf{T}^k\mathsf{T}^l \arrow[dl, "\tau_{k\mathsf{T}^l}"'] \arrow[dr, "\mathsf{T}^k\tau_l"] & \\ \mathsf{T}^l \arrow[dr, "\tau_l"'] & & \mathsf{T}^k \arrow[dl, "\tau_k"] \\ & \mathsf{I} & \end{tikzcd}$$

We have observed that for each manifold M the space $\mathsf{T}^k\mathsf{T}^l M$ can be identified with the space $\mathsf{T}^{(k,l)}M$. We use mappings $\chi : \mathbb{R}^2 \to M$ to represent elements of $\mathsf{T}^k\mathsf{T}^l M$. In terms of this representation we define the mapping

$$\kappa^{(k,l)}{}_M : \mathsf{T}^k\mathsf{T}^l M \to \mathsf{T}^l\mathsf{T}^k M : \mathsf{t}^{(k,l)}\chi(0,0) \mapsto \mathsf{t}^{(l,k)}\widetilde{\chi}(0,0),$$

where $\widetilde{\chi}$ is the mapping

$$\widetilde{\chi} : \mathbb{R}^2 \to M : (t,s) \mapsto \chi(s,t).$$

The result of this construction is the natural transformation

$$
\begin{tikzcd}
& \mathsf{T}^k\mathsf{T}^l \arrow[rrr, "\kappa^{(k,l)}"] \arrow[dl, "\tau_{k\mathsf{T}^l}"'] \arrow[ddr, "\mathsf{T}^k\tau_l"] & & & \mathsf{T}^l\mathsf{T}^k \arrow[dl, "\mathsf{T}^l\tau_k"'] \arrow[ddr, "\tau_{l\mathsf{T}^k}"] & \\
\mathsf{T}^l \arrow[rrr, equal] \arrow[ddr, "\tau_l"'] & & & \mathsf{T}^l \arrow[ddr, "\tau_l"'] & & \\
& & \mathsf{T}^k \arrow[rrr, equal] \arrow[dl, "\tau_k"'] & & & \mathsf{T}^k \arrow[dl, "\tau_k"'] \\
& \mathsf{I} \arrow[rrr, equal] & & & \mathsf{I} &
\end{tikzcd}
\quad .
$$

The diagram

$$
\begin{tikzcd}
& \mathsf{T}^k\mathsf{T}^l \arrow[rrr, "\kappa^{(k,l)}"] \arrow[dl, "{\tau^{k'}}_{k\mathsf{T}^l}"'] \arrow[ddr, "\mathsf{T}^k{\tau^{l'}}_l"] & & & \mathsf{T}^l\mathsf{T}^k \arrow[dl, "\mathsf{T}^l{\tau^{k'}}_k"'] \arrow[ddr, "{\tau^{l'}}_{l\mathsf{T}^k}"] & \\
\mathsf{T}^{k'}\mathsf{T}^l \arrow[rrr, "\kappa^{(k',l)}"] \arrow[ddr, "\mathsf{T}^{k'}{\tau^{l'}}_l"'] & & & \mathsf{T}^l\mathsf{T}^{k'} \arrow[ddr, "{\tau^{l'}}_{l\mathsf{T}^{k'}}"'] & & \\
& & \mathsf{T}^k\mathsf{T}^{l'} \arrow[rrr, "\kappa^{(k,l')}"] \arrow[dl, "{\tau^{k'}}_{k\mathsf{T}^{l'}}"'] & & & \mathsf{T}^{l'}\mathsf{T}^k \arrow[dl, "\mathsf{T}^{l'}{\tau^{k'}}_k"'] \\
& \mathsf{T}^{k'}\mathsf{T}^{l'} \arrow[rrr, "\kappa^{(k',l')}"] & & & \mathsf{T}^{l'}\mathsf{T}^{k'} &
\end{tikzcd}
$$

expresses properties of the natural transformation $\kappa^{(k,l)}$ represented also in the two following diagrams.

$$
\begin{tikzcd}
\mathsf{T}^k\mathsf{T}^l M \arrow[r, "{\kappa^{(k,l)}}_M"] \arrow[d, "{\tau^{k'}}_{k\mathsf{T}^l M}"'] & \mathsf{T}^l\mathsf{T}^k M \arrow[d, "\mathsf{T}^l{\tau^{k'}}_{kM}"'] \\
\mathsf{T}^{k'}\mathsf{T}^l M \arrow[r, "{\kappa^{(k',l)}}_M"] & \mathsf{T}^l\mathsf{T}^{k'} M
\end{tikzcd}
$$

for $k' \leqslant k$ and

$$
\begin{tikzcd}
\mathsf{T}^k\mathsf{T}^l M \arrow[r, "{\kappa^{(k,l)}}_M"] \arrow[d, "\mathsf{T}^k{\tau^{l'}}_{lM}"'] & \mathsf{T}^l\mathsf{T}^k M \arrow[d, "{\tau^{l'}}_{l\mathsf{T}^k M}"'] \\
\mathsf{T}^{k'}\mathsf{T}^l M \arrow[r, "{\kappa^{(k',l)}}_M"] & \mathsf{T}^l\mathsf{T}^{k'} M
\end{tikzcd}
$$

for $l' \leqslant l$.

Relations

$$
{\kappa^{(k,l)}}_M \circ {\kappa^{(l,k)}}_M = 1_{\mathsf{T}^l\mathsf{T}^k M}
$$

are obviously satisfied. The special case $\kappa_M = \kappa^{(1,1)}{}_M$ is the most frequently used. It is known as the canonical involution in $\mathsf{T}\mathsf{T}M = \mathsf{T}^1\mathsf{T}^1M$.

We introduce mappings

$$\lambda^{(k,l)}{}_M : \mathsf{T}^{k+l}M \to \mathsf{T}^k\mathsf{T}^lM : \mathsf{t}^{k+l}\gamma(0) \mapsto \mathsf{t}^{(k,l)}\chi(0,0),$$

where χ is the mapping

$$\chi : \mathbb{R}^2 \to M : (s,t) \mapsto \gamma(s+t).$$

In this definition we are using the identification of $\mathsf{T}^k\mathsf{T}^lM$ with $\mathsf{T}^{(k,l)}M$. An alternative definition is given by

$$\lambda^{k,l}{}_M : \mathsf{T}^{k+l}M \to \mathsf{T}^k\mathsf{T}^lM : \mathsf{t}^{k+l}\gamma(0) \mapsto \mathsf{t}^k\mathsf{t}^l\gamma(0),$$

The commutative diagram

$$\begin{tikzcd}
 & \mathsf{T}^{k+l} \arrow[dl, "\tau^l{}_{k+l}"'] \arrow[ddr, "\tau^k{}_{k+1}"] \arrow[rrr, "\lambda^{(k,l)}"] & & & \mathsf{T}^k\mathsf{T}^l \arrow[dl, "\tau_{k\mathsf{T}^l}"'] \arrow[ddr, "\mathsf{T}^k\tau_l"] & \\
\mathsf{T}^l \arrow[rrr, equal] \arrow[ddr, "\tau_l"'] & & & \mathsf{T}^l \arrow[ddr, "\tau_l"'] & & \\
 & & \mathsf{T}^k \arrow[rrr, equal] \arrow[dl, "\tau_k"'] & & & \mathsf{T}^k \arrow[dl, "\tau_k"'] \\
 & \mathsf{I} \arrow[rrr, equal] & & & \mathsf{I} &
\end{tikzcd}.$$

presents $\lambda^{(k,l)}$ as a natural transformation.

Mappings $\lambda^k{}_M = \lambda^{(1,k)}{}_M$ and $\lambda_M = \lambda^{(1,1)}{}_M$ are of particular interest.

6. Applications to the calculus of variations [6] [7].

6.1. Derivations.

Let $\Omega(M)$ be the exterior algebra of differential forms on a differential manifold M. A linear operator $\mathrm{a} : \Omega(M) \to \Omega(M)$ is called a *derivation* of $\Omega(M)$ of degree p if $\mathrm{a}\mu$ is a form of degree $q+p$ and

$$\mathrm{a}(\mu \wedge \nu) = \mathrm{a}\mu \wedge \nu + (-1)^{pq}\mu \wedge \mathrm{a}\nu$$

when μ is a form of degree q and ν is any form on M. The exterior differential $\mathrm{d} : \Omega(M) \to \Omega(M)$ is a derivation of degree 1. The *commutator*

$$[\mathrm{a}, \mathrm{a}'] = \mathrm{a}\mathrm{a}' - (-1)^{pp'}\mathrm{a}'\mathrm{a}$$

of derivations a and a' of degrees p and p' respectively is a derivation of degree $p+p'$. A derivation a is said to be of *type* i_* if $\mathrm{a}f = 0$ for each function f on M. A derivation a is said to be of *type* d_* if $[\mathrm{a}, \mathrm{d}] = 0$. If i_A is a derivation of type i_*, then $\mathrm{d}_A = [\mathrm{i}_A, \mathrm{d}]$ is a derivation of type d_*. Derivations are local operators: if a is a derivation and μ is a differential form on M vanishing on an open subset $U \subset M$, then $\mathrm{a}\mu$ vanishes on U. A derivation is fully characterized by its action on functions and differentials of functions since each differential form is locally representable as a sum of exterior products of differentials of

functions multiplied by functions. A derivation of type d_* is fully characterized by its action on functions.

6.2. Vector-valued forms and derivations.

A *vector-valued p-form* is a linear mapping

$$A : \wedge^p \mathsf{T}M \to \mathsf{T}M.$$

If $w \in \wedge^p \mathsf{T}_x M$, then $A(w) \in \mathsf{T}_x M$. Following Frölicher and Nijenhuis [2] we associate with a vector-valued p-form A a derivation i_A of type i_* and degree $p-1$ and the derivation $\mathrm{d}_A = [\mathrm{i}_A, \mathrm{d}]$. The derivation i_A is characterized by its action on 1-forms. If μ is a 1-form, then $\mathrm{i}_A\mu$ is a p-form and

$$\langle \mathrm{i}_A\mu, w\rangle = \langle \mu, A(w)\rangle$$

for each $w \in \wedge^p \mathsf{T}M$.

For each $k \in \mathbb{N}$ and each $n \in \mathbb{N}$ we define a linear mapping

$$F(k;n) : \mathsf{T}\mathsf{T}^k M \to \mathsf{T}\mathsf{T}^k M : \mathsf{t}^{(1,k)}\chi(0,0) \mapsto \mathsf{t}^{(1,k)}\chi^n(0,0),$$

where χ is a mapping from $\mathbb{R}^2$ to M and

$$\chi^n : \mathbb{R}^2 \to M : (s,t) \mapsto \chi(st^n, t).$$

Relations

$$F(k;0) = 1_{\mathsf{T}\mathsf{T}^k M},$$
$$F(k;n') \circ F(k;n) = F(k;n'+n),$$

and

$$F(k;n) = 0 \quad \text{if } n \geqslant k$$

are easily established. The diagram

$$\begin{tikzcd}
\mathsf{T}\mathsf{T}^k M \arrow[r, "F(k;n)"] \arrow[d, "\tau_{\mathsf{T}^* M}"'] & \mathsf{T}\mathsf{T}^k M \arrow[d, "\tau_{\mathsf{T}^* M}"'] \\
\mathsf{T}^k M \arrow[r, equal] & \mathsf{T}^k M
\end{tikzcd}$$

is commutative since $\chi^n(0,\cdot) = \chi(0,\cdot)$ and the diagram

$$\begin{tikzcd}
\mathsf{T}\mathsf{T}^k M \arrow[r, "F(k;n)"] \arrow[d, "\mathsf{T}\tau^{k'}{}_{kM}"'] & \mathsf{T}\mathsf{T}^k M \arrow[d, "\mathsf{T}\tau^{k'}{}_{kM}"'] \\
\mathsf{T}\mathsf{T}^{k'} M \arrow[r, "F(k';n)"] & \mathsf{T}\mathsf{T}^{k'} M
\end{tikzcd}$$

is obvously commutative. It follows that the mappings $F(k;n)$ are vector-valued 1-forms.

Let $\Omega_k(M)$ denote the exterior algebra of differential forms on the k-jet bundle $\mathsf{T}^k M$. We will denote by ${\omega_k}^{k'}{}_M$ the homomorphism

$$\tau^{k'}{}_{kM}{}^{*} : \Omega_{k'}(M) \to \Omega_k(M).$$

Derivations $\mathrm{i}_{F(k;n)}$ and $\mathrm{d}_{F(k;n)}$ are associated with the vector-valued 1-forms $F(k;n)$. Diagrams

$$\begin{tikzcd}
\Omega_{k'}(M) \arrow[r, "\mathrm{i}_{F(k';n)}"] \arrow[d, "{\omega_k}^{k'}{}_M"'] & \Omega_{k'}(M) \arrow[d, "{\omega_k}^{k'}{}_M"'] \\
\Omega_k(M) \arrow[r, "\mathrm{i}_{F(k;n)}"] & \Omega_k(M)
\end{tikzcd}$$

are commutative.

6.3. Generalized vector-valued forms and derivations.

The article [3] offers a generalization of the Frölicher and Nijenhuis theory. Let $\varphi : N \to M$ be a differentiable mapping. The mapping $\varphi^* : \Omega(M) \to \Omega(N)$ is a homomorphism of the exterior algebras. A *derivation of degree p relative to φ^** is a linear operator $\mathrm{a} : \Omega(M) \to \Omega(N)$ such that $\mathrm{a}\mu$ is a form on N of degree $q + p$ and

$$\mathrm{a}(\mu \wedge \nu) = \mathrm{a}\mu \wedge \varphi^*\nu + (-1)^{pq}\varphi^*\mu \wedge \mathrm{a}\nu$$

if μ is a form on M of degree q and ν is any form on M. A derivation of the algebra $\Omega(M)$ is a derivation relative to the identity mapping 1_M. A derivation a relative to φ is said to be of *type* i_* if $\mathrm{a}f = 0$ for each function f on M. A relative derivation a of degree p is said to be of *type* d_* if $\mathrm{ad} - (-1)^p\mathrm{da} = 0$. If i_A is a derivation of type i_* relative to φ, then $\mathrm{d}_A = \mathrm{i}_A\mathrm{d} - (-1)^p\mathrm{d}\mathrm{i}_A$ is a derivation of type d_* relative to φ. Note that the expressions $\mathrm{ad} - (-1)^p\mathrm{da}$ and $\mathrm{i}_A\mathrm{d} - (-1)^p\mathrm{d}\mathrm{i}_A$ are not commutators since each of these expressions involves two different exterior differentials d. If a is a derivation of degree p relative to φ^* and $\psi : O \to N$ is a differentiable mapping, then the operator $\psi^*\mathrm{a} : \Omega(M) \to \Omega(O)$ is a derivation of degree p relative to $(\varphi \circ \psi)^*$ since

$$\begin{aligned}
\psi^*\mathrm{a}(\mu \wedge \nu) &= \psi^*\mathrm{a}\mu \wedge \psi^*\varphi^*\nu + (-1)^{pq}\psi^*\varphi^*\mu \wedge \psi^*\mathrm{a}\nu \\
&= \psi^*\mathrm{a}\mu \wedge (\varphi \circ \psi)^*\nu + (-1)^{pq}(\varphi \circ \psi)^*\mu \wedge \psi^*\mathrm{a}\nu
\end{aligned}$$

if μ is a form on M of degree q and ν is any form on M. If a is a derivation of type i_* or d_*, then $\psi^*\mathrm{a}$ is a derivation of the same type. Relative derivations are again local operators and are completely characterized by their action on functions and differentials of functions.

A *vector-valued p-form relative to $\varphi : N \to M$* is a linear mapping

$$A : \wedge^p \mathsf{T}N \to \mathsf{T}M$$

such that if $w \in \wedge^p \mathsf{T}_b N$, then $A(w) \in \mathsf{T}_{\varphi(b)}M$. We associate with a vector-valued p-form A relative to φ a derivation i_A relative to φ^* of type i_* and degree $p - 1$ and the relative

derivation $\mathrm{d}_A = \mathrm{i}_A\mathrm{d} - (-1)^p\mathrm{d}\mathrm{i}_A$. If μ is a 1-form on M, then $\mathrm{i}_A\mu$ is a p-form on N and

$$\langle \mathrm{i}_A\mu, w\rangle = \langle \mu, A(w)\rangle$$

for each $w \in \wedge^p\mathsf{T}N$.

For each $k \in \mathbb{N}$ we introduce the mapping

$$T(k)\colon \mathsf{T}^{k+1}M \to \mathsf{T}\mathsf{T}^kM : \mathsf{t}^{k+1}\gamma(0) \mapsto \mathsf{t}\mathsf{t}^k\gamma(0).$$

For $k = 0$ we have

$$T(0)\colon \mathsf{T}M \to \mathsf{T}M : \mathsf{t}^1\gamma(0) \mapsto \mathsf{t}\gamma(0).$$

Diagrams

$$\begin{tikzcd} \mathsf{T}^{k+1}M \arrow[r, "T(k)"] \arrow[d, "{\tau^{k'+1}}_{k+1M}"'] & \mathsf{T}\mathsf{T}^kM \arrow[d, "\mathsf{T}{\tau^{k'}}_{kM}"] \\ \mathsf{T}^{k'+1}M \arrow[r, "T(k')"] & \mathsf{T}\mathsf{T}^{k'M} \end{tikzcd}$$

are commutative.

Interpreting the mapping $T(k)$ as a vector-valued 0-form relative to

$${\tau^k}_{k+1M}\colon \mathsf{T}^{k+1}M \to \mathsf{T}^kM$$

we introduce derivations $\mathrm{i}_{T(k)}\colon \Omega_k(M) \to \Omega_{k+1}(M)$ and $\mathrm{d}_{T(k)}\colon \Omega_k(M) \to \Omega_{k+1}(M)$ relative to ${\omega_{k+1}}^k{}_{(M)}$. The derivation $\mathrm{i}_{T(k)}$ is a derivation of degree -1. The derivation $\mathrm{d}_{T(k)} = \mathrm{i}_{T(k)}\mathrm{d} + \mathrm{d}\mathrm{i}_{T(k)}$ of degree 0 is known in the calculus of variations as the *total derivative.* Diagrams

$$\begin{tikzcd} \Omega_{k'}(M) \arrow[r, "\mathrm{i}_{T(k')}"] \arrow[d, "{\omega_k}^{k'}{}_M"'] & \Omega_{k'+1}(M) \arrow[d, "{\omega_{k+1}}^{k'+1}{}_M"] \\ \Omega_k(M) \arrow[r, "\mathrm{i}_{T(k)}"] & \Omega_{k+1}(M) \end{tikzcd}$$

and

$$\begin{tikzcd} \Omega_{k'}(M) \arrow[r, "\mathrm{d}_{T(k')}"] \arrow[d, "{\omega_k}^{k'}{}_M"'] & \Omega_{k'+1}(M) \arrow[d, "{\omega_{k+1}}^{k'+1}{}_M"] \\ \Omega_k(M) \arrow[r, "\mathrm{d}_{T(k)}"] & \Omega_{k+1}(M) \end{tikzcd}$$

are commutative.

6.4. The Euler-Lagrange differential.

Let M be manifold. A *parameterized differentiable arc* is the restriction $\xi|[t_0,t_1]$ of an emedding $\xi\colon\mathbb{R}\to M$ to an interval $[t_0,t_1]\subset\mathbb{R}$. The space $\mathsf{Q}(M|\mathbb{R})$ of arcs is not a differential manifold.

In terms of differentiable homotopies $\chi\colon\mathbb{R}^2\to M$ we define curves

$$\gamma|[t_0,t_1]\colon\mathbb{R}\to\mathsf{Q}(M|\mathbb{R})\colon s\mapsto\chi(s,\cdot)|[t_0,t_1]$$

considered differentiable by definition.

Let $L\colon\mathsf{T}^kM\to\mathbb{R}$ be a differentiable function. The mapping

$$A\colon\mathsf{Q}(M|\mathbb{R})\to\mathbb{R}\colon\xi|[t_0,t_1]\mapsto\int_{t_0}^{t_1}L\circ\mathsf{t}^k\xi$$

is considered a differentiable function. There is a natural projection

$$Pr\colon\mathsf{Q}(M|\mathbb{R})\to\mathsf{T}^{k-1}M\times\mathsf{T}^{k-1}M\colon\xi|[t_0,t_1]\mapsto(\mathsf{t}^{k-1}\xi(t_0),\mathsf{t}^{k-1}\xi(t_1)).$$

The function A is considered a family of functions defined on fibres of this projection and the calculus of variations is a study of the critical set of this family. An arc $\xi|[t_0,t_1]$ is a *critical arc* for the family if

$$\mathrm{D}(A\circ\gamma|[t_0,t_1])(0)=0$$

for each homotopy $\chi\colon\mathbb{R}^2\to M$ such that $\chi(0,\cdot)=\xi$, $\mathsf{t}^{k-1}\chi(s,\cdot)(t_0)=\mathsf{t}^{k-1}\xi(t_0)$, and $\mathsf{t}^{k-1}\chi(s,\cdot)(t_1)=\mathsf{t}^{k-1}\xi(t_1)$ for each s. The curve $\gamma|[t_0,t_1]$ derived from such homotopy is said to be *vertical*.

By introducing mappings

$$\delta\xi\colon\mathbb{R}\to\mathsf{T}M\colon t\mapsto\mathsf{t}\chi(\cdot,t)(0)$$

and

$$\delta\mathsf{t}^k\xi=\kappa^{(k,1)}{}_M\circ\mathsf{t}^k\delta\xi$$

we obtain the expression

$$\begin{aligned}\mathrm{D}(A\circ\gamma|[t_0,t_1])(0)&=\frac{\mathrm{d}}{\mathrm{d}s}\int_{t_0}^{t_1}L\circ\mathsf{t}^k\chi(s,\cdot)\big|_{s=0}\\&=\int_{t_0}^{t_1}\left\langle\mathrm{d}L,\delta\mathsf{t}^k\xi\right\rangle.\end{aligned}$$

This expression is converted to an equivalent expression

$$\begin{aligned}\mathrm{D}(A\circ\gamma|[t_0,t_1])(0)&=\int_{t_0}^{t_1}\langle\mathrm{d}L,\delta\mathsf{t}^k\xi\rangle\\&=\int_{t_0}^{t_1}\langle\mathrm{d}L,\mathsf{T}\tau^k{}_{2kM}\circ\delta\mathsf{t}^{2k}\xi\rangle\\&=\int_{t_0}^{t_1}\langle\omega_{2k}{}^k{}_M\mathrm{d}L,\delta\mathsf{t}^{2k}\xi\rangle\\&=\int_{t_0}^{t_1}\langle E(k)\mathrm{d}L,\delta\mathsf{t}^{2k}\xi\rangle-\int_{t_0}^{t_1}\langle\mathrm{d}_{T(2k-1)}P(k)\mathrm{d}L,\delta\mathsf{t}^{2k}\xi\rangle\end{aligned}$$

with operators

$$E(k)\colon \Omega_k(M) \to \Omega_{2k}(M),$$

and

$$P(k)\colon \Omega_k(M) \to \Omega_{2k-1}(M)$$

defined by

$$E(k) = \sum_{n=0}^{k} \frac{(-1)^n}{n!} {\omega_{2k}}^{k+n}{}_M \mathrm{d}_{T(k+n-1)} \mathrm{d}_{T(k+n-2)} \cdots \mathrm{d}_{T(k)} \mathrm{i}_{F(k;n)}.$$

$$P(0) = 0,$$

and

$$P(k) = \sum_{n=1}^{k} \frac{(-1)^n}{n!} {\omega_{2k-1}}^{k+n-1}{}_M \mathrm{d}_{T(k+n-2)} \cdots \mathrm{d}_{T(k)} \mathrm{i}_{F(k;n)}$$

for $k > 0$.

The form $E(k)\mathrm{d}L \in \Omega_{2k}(M)$ is vertical with respect to the projection

$$\tau_{2k\,M}\colon \mathsf{T}^{2k}M \to M$$

and the form $P(k)\mathrm{d}L \in \Omega_{2k-1}(M)$ is vertical with respect to the projection

$$\tau^{k-1}{}_{2k-1\,M}\colon \mathsf{T}^{2k-1}M \to \mathsf{T}^{k-1}M.$$

Verticality of these forms makes it possible to define mappings

$$\mathcal{E}L\colon \mathsf{T}^{2k} \to \mathsf{T}^*M$$

and

$$\mathcal{P}L\colon \mathsf{T}^{2k-1}M \to \mathsf{T}^*\mathsf{T}^{k-1}M$$

such that

$$\pi_M \circ \mathcal{E}L = \tau_{2k\,M}$$

and

$$\pi_M \circ \mathcal{P}L = \tau_{2k-1\,M}.$$

These mappings are characterized by

$$\langle E(k)\mathrm{d}L, w\rangle = (-1)^k \langle \mathcal{E}L(\tau_{\mathsf{T}^{2k}M}(w)), \mathsf{T}\tau_{2k\,M}(w)\rangle$$

for each $w \in \mathsf{T}\mathsf{T}^{2k}M$ and

$$\langle P(k)\mathrm{d}L, w\rangle = \langle \mathcal{P}L(\tau_{\mathsf{T}^{2k-1}M}(w)), \mathsf{T}\tau^{k-1}{}_{2k-1\,M}(w)\rangle$$

for each $w \in \mathsf{T}\mathsf{T}^{2k-1}M$.

The following final expression for the derivative of A is obtained:

$$\int_{t_0}^{t_1} \langle \mathcal{E}L \circ \mathsf{t}^{2k}\xi, \delta\xi\rangle - \langle (\mathcal{P}L \circ \mathsf{t}^{2k-1}\xi)(t_1), \delta\mathsf{t}^{k-1}\xi(t_1)\rangle + \langle (\mathcal{P}L \circ \mathsf{t}^{2k-1}\xi)(t_0), \delta\mathsf{t}^{k-1}\xi(t_0)\rangle.$$

If the curve $\gamma|[t_0, t_1]$ is vertical, then the boundary terms vanish and the *Euler-Lagrange equation*

$$\mathcal{E}L \circ \mathsf{t}^{2k}\xi = 0$$

is obtained as the condition for the arc $\xi|[t_0, t_1]$ to be critical. The boundary terms are important in variational principles of physics but not in the calculus of variations.

7. A framework for the Legendre transformation [8] [9] [10].

The diagram

$$\begin{tikzcd}
\mathsf{T}^*\mathsf{T}^*M \arrow[dr, "\pi_{\mathsf{T}^*M}"'] & & \mathsf{TT}^*M \arrow[ll, "\beta_{(\mathsf{T}^*M,\omega_M)}"'] \arrow[rr, "\alpha_M"] \arrow[dl, "\tau_{\mathsf{T}^*M}"] \arrow[dr, "\mathsf{T}\pi_M"'] & & \mathsf{T}^*\mathsf{T}M \arrow[dl, "\pi_{\mathsf{T}M}"] \\
& \mathsf{T}^*M \arrow[dr, "\pi_M"'] & & \mathsf{T}M \arrow[dl, "\tau_M"] & \\
& & M & &
\end{tikzcd}$$

contains the essential geometric objects used in the formulation of the Legendre transformation of analytical mechanics.

On the Hamiltonian side we have the vector fibration isomorphism

$$\begin{tikzcd}
\mathsf{TT}^*M \arrow[r, "\beta_{(\mathsf{T}^*M,\omega_M)}"] \arrow[d, "\tau_{\mathsf{T}^*M}"'] & \mathsf{T}^*\mathsf{T}^*M \arrow[d, "\pi_{\mathsf{T}^*M}"] \\
\mathsf{T}^*M \arrow[r, equal] & \mathsf{T}^*M
\end{tikzcd}$$

derived from the symplectic structure of T^*M.

On the Lagrangian side there is the vector fibration isomorphism

$$\begin{tikzcd}
\mathsf{TT}^*M \arrow[r, "\alpha_M"] \arrow[d, "\mathsf{T}\pi_M"'] & \mathsf{T}^*\mathsf{T}M \arrow[d, "\pi_{\mathsf{T}M}"] \\
\mathsf{T}M \arrow[r, equal] & \mathsf{T}M
\end{tikzcd}$$

defined as dual to the vector fibration isomorphism

$$\begin{tikzcd} \mathsf{TT}M \arrow[d, "\mathsf{T}\tau_M"'] & & \mathsf{TT}M \arrow[ll, "\kappa_M"'] \arrow[d, "\tau_{\mathsf{T}M}"] \\ \mathsf{T}M \arrow[rr, equal] & & \mathsf{T}M \end{tikzcd}$$

in the sense that

$$\langle \alpha_M(z), w\rangle = \langle z, \kappa_M(w)\rangle$$

for $z \in \mathsf{TT}^*M$ and $w \in \mathsf{TT}M$ such that $\mathsf{T}\pi_M(z) = \tau_{\mathsf{T}M}(w)$.

8. Isomorphisms between TT^*M, $\mathsf{T}^*\mathsf{T}M$, and $\mathsf{T}^*\mathsf{T}^*M$.

We introduce the set $\mathsf{W}(M)$ of equivalence classes of pairs (f, γ) of a function $f : \mathbb{R} \times M \to \mathbb{R}$ and a curve $\gamma : \mathbb{R} \to M$. The equivalence is defined in terms of functions

$$F : \mathbb{R}^2 \to \mathbb{R} : (s, t) \mapsto f(s, \varphi(t, \gamma(s)))$$

and

$$F' : \mathbb{R}^2 \to \mathbb{R} : (s, t) \mapsto f'(s, \varphi(t, \gamma'(s)))$$

associated with pairs (f, γ) and (f', γ'), and a mapping $\varphi : \mathbb{R} \times M \to M$ such that $\varphi(0, x) = x$ for each $x \in M$. Pairs (f, γ) and (f', γ') are equivalent if

$$\mathsf{t}\gamma'(0) = \mathsf{t}\gamma(0),$$
$$\mathrm{d}f'(0, \cdot)(\gamma(0)) = \mathrm{d}f(0, \cdot)(\gamma(0)),$$

and

$$\mathrm{D}^{(1,1)} F'(0, 0) = \mathrm{D}^{(1,1)} F(0, 0).$$

Coordinates

$$(x^\kappa, \dot{x}^\lambda, a_\mu, b_\nu) : \mathsf{W}M \to \mathbb{R}^{4m}$$

in $\mathsf{W}(M)$ are defined by

$$x^\kappa(w) = x^\kappa(\gamma(0)),$$
$$\dot{x}^\lambda(w) = \mathrm{D}(x^\lambda \circ \gamma)(0),$$
$$a_\mu(w) = \partial_\mu \overline{f}(\cdot)(\gamma(0))$$

with

$$\overline{f} : M \to \mathbb{R} : x \mapsto \mathrm{D}f(\cdot, x)(0),$$

and

$$b_\nu(w) = \partial_\nu f(0, \cdot)(\gamma(0)).$$

Given coordinates $(x^\kappa(w), \dot{x}^\lambda(w), a_\mu(w), b_\nu(w))$ of an element $w \in \mathsf{W}(M)$ we construct a representative (f, γ) of w. The curve γ is characterized by

$$(x^\kappa \circ \gamma)(s) = x^\kappa(w) + \dot{x}^\kappa(w)s$$

and the function f is defined by

$$f(s,x) = b_\nu(w)x^\nu(x) + a_\mu(w)x^\mu(x)s.$$

We establish an isomorphism of $\mathsf{W}(M)$ with TT^*M. A pair (f,γ) representing an element $w \in \mathsf{W}(M)$ is used to construct a curve

$$\rho\colon \mathbb{R} \to \mathsf{T}^*M : s \mapsto \mathrm{d}f(s,\cdot)(\gamma(s)).$$

The tangent vector $\mathsf{t}\rho(0)$ is the element of TT^*M associated with w.

We establish an isomorphism of $\mathsf{W}(M)$ with $\mathsf{T}^*\mathsf{T}M$. A pair (f,γ) representing an element $w \in \mathsf{W}(M)$ is used to construct a function

$$g\colon \mathsf{T}M \to \mathbb{R} : \mathsf{t}\sigma(0) \mapsto \mathrm{D}f(\cdot,\sigma(\cdot))(0).$$

The covector $\mathrm{d}g(x)$ at $x = \mathsf{t}\gamma(0)$ is the element of $\mathsf{T}^*\mathsf{T}M$ associated with w.

We establish an isomorphism of $\mathsf{W}(M)$ with $\mathsf{T}^*\mathsf{T}^*M$. A pair (f,γ) representing an element $w \in \mathsf{W}(M)$ is used to construct a function

$$h\colon \mathsf{T}^*M \to \mathbb{R} : \mathrm{d}k(x) \mapsto \mathrm{D}(f(\cdot,x) - k \circ \gamma)(0).$$

The covector $\mathrm{d}h(p)$ at $p = \mathrm{d}f(0,\cdot)(\gamma(0))$ is the element of $\mathsf{T}^*\mathsf{T}^*M$ associated with w.

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